

Mean link versus average plaquette tadpoles in lattice NRQCD

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We compare mean-link and average plaquette tadpole renormalization schemes in the context of the quarkonium hyperfine splittings in lattice NRQCD. Simulations are done for the three quarkonium systems $c\bar{c}$, $b\bar{c}$, and $b\bar{b}$. The hyperfine splittings are computed both at leading and at next-to-leading order in the relativistic expansion. Results are obtained at a large number of lattice spacings. A number of features emerge, all of which favor tadpole renormalization using mean links. This includes much better scaling of the hyperfine splittings in the three quarkonium systems. We also find that relativistic corrections to the spin splittings are smaller with mean-link tadpoles, particularly for the $c\bar{c}$ and $b\bar{c}$ systems. We also see signs of a breakdown in the NRQCD expansion when the bare quark mass falls below about one in lattice units (with the bare quark masses turning out to be much larger with mean-link tadpoles).

Tadpole diagrams in lattice theories are induced by the nonlinear connection between the lattice link variables U_μ and the continuum gauge fields. This causes large radiative corrections to many quantities in lattice theories. Most of the effects of tadpoles can be removed by a mean field renormalization of the links [1]

$$U_\mu(x) \rightarrow \frac{U_\mu(x)}{u_0}, \quad (1)$$

where an operator dominated by short-distance fluctuations is used to determine u_0 .

One of the earliest applications of tadpole improvement was in the development of lattice non-relativistic quantum chromodynamics (NRQCD) [2–7]. Precision simulations of the Υ system in NRQCD have provided important phenomenological results, including the strong coupling constant [4] and the b -quark pole mass [5,6]. However the situation for charmonium is more problematic, due to large relativistic corrections [8].

The quarkonium spectrum provides a powerful probe of tadpole renormalization. The quarkonium hyperfine spin splittings in particular are very sensitive to the details of the NRQCD Hamiltonian, with the relevant operators undergoing large tadpole renormalizations. For example, it has been shown [8] that scaling of the charmonium hyperfine splitting is significantly improved

when the tadpole renormalization is determined using the mean-link $u_{0,L}$ measured in Landau gauge [1,9–11]:

$$u_{0,L} \equiv \langle \text{ReTr } U_\mu \rangle, \quad \partial_\mu A_\mu = 0, \quad (2)$$

compared to when the fourth root of the average plaquette $u_{0,P}$ is used:

$$u_{0,P} \equiv \langle \text{ReTr } U_{\text{pl}} \rangle^{1/4}. \quad (3)$$

We make a comparison of the two tadpole renormalization schemes $u_{0,L}$ and $u_{0,P}$ (implemented at tree-level) in the context of the quarkonium hyperfine splittings in NRQCD. This is done for the three quarkonium systems $c\bar{c}$, $b\bar{c}$, and $b\bar{b}$. The hyperfine splittings are computed both at leading ($O(M_Q v^4)$) and at next-to-leading ($O(M_Q v^6)$) order in the relativistic expansion. (For further details see Ref. [12])

All quantities are calculated after re-tuning of the lattice action parameters for each system. The resulting quark masses and lattice spacings for the three quarkonium systems for the NRQCD action at $O(v^6)$ are given in Tables 1 and 2. To minimize systematic errors from quenching the b -quark mass is tuned separately to 1S_0 for the $b\bar{c}$ and $b\bar{b}$ states with the c -quark mass tuned to the 1S_0 $c\bar{c}$ state. Likewise, the lattice spacings were determined separately from the spin-averaged $1P-1S$ mass difference, which we set to 458 MeV (experimental value for charmonium).

Table 1

Bare quark masses for the three quarkonium systems at $O(v^6)$, using Landau gauge mean-link tadpoles $u_{0,L}$; the stability parameter n for each mass is given in square brackets.

β_L	$a_{c\bar{c}}$ (fm)	$aM_c^0[n]$	$aM_b^0[n]$	$aM_b^0[n]$
7.5	0.155(4)	1.10[4]	3.20[2]	3.20[2]
7.4	0.179(2)	1.20[4]	3.57[2]	3.57[2]
7.0	0.280(4)	1.97[2]	6.10[2]	5.35[2]
6.85	0.319(5)	2.25[2]	6.50[2]	5.90[2]
6.7	0.361(6)	2.50[2]	7.20[2]	6.35[2]
6.6	0.380(7)	2.67[2]	7.50[2]	6.66[2]

Table 2

Bare quark masses for the quarkonium systems at $O(v^6)$ using average plaquette tadpoles $u_{0,P}$.

β_P	$a_{c\bar{c}}$ (fm)	$aM_c^0[n]$	$aM_b^0[n]$	$aM_b^0[n]$
7.3	0.140(4)	0.65[8]	2.87[2]	2.87[2]
7.2	0.169(2)	0.83[4]	3.20[2]	3.20[2]
7.0	0.210(2)	1.10[4]	4.10[2]	3.95[2]
6.8	0.256(3)	1.43[3]	4.98[2]	4.53[2]
6.6	0.313(4)	1.80[3]	5.83[2]	5.23[2]
6.4	0.350(6)	2.15[2]	6.45[2]	5.60[2]
6.25	0.390(6)	2.41[2]	6.85[2]	5.99[2]

The lattice NRQCD effective action for quarkonium is organized according to an expansion in the mean squared velocity v^2 of the heavy quarks, with corrections included for lattice artifacts. The effective action, including spin-independent operators to $O(v^4)$, and spin-dependent interactions to $O(v^6)$, was derived in Ref. [3]. Following Refs. [6,7], we use the evolution equation

$$G_{t+1} = \left(1 - \frac{aH_0}{2n}\right)^n U_4^\dagger \left(1 - \frac{aH_0}{2n}\right)^n (1 - a\delta H) G_t. \quad (4)$$

Relativistic corrections are organized in powers of the heavy quark velocity:

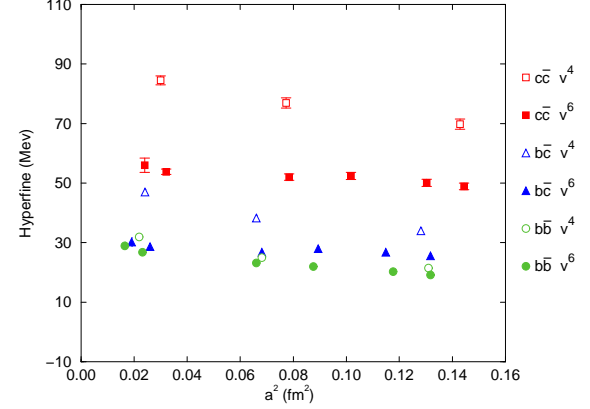
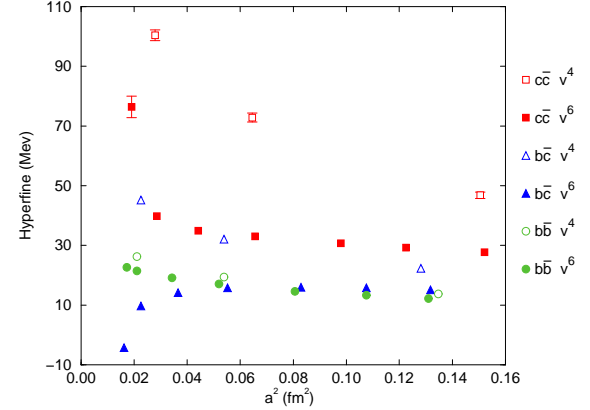
$$\delta H = \delta H^{(4)} + \delta H^{(6)}. \quad (5)$$

Only next-to-leading spin-dependent interactions are considered in $\delta H^{(6)}$.

Simulations were done with the derivative operators and the clover fields corrected for their leading discretization errors. Complete expressions for the operators can be found in Refs. [3,8,12].

Gauge-invariant source and sink smearing was used for Meson operators.

The gauge-field configurations were made using a tree-level $O(a^4)$ -accurate tadpole-improved action [13]

Figure 1. Hyperfine splittings with $u_{0,L}$ versus lattice spacing squared.Figure 2. Hyperfine splittings with $u_{0,P}$ versus lattice spacing squared.

There are a number of clear features in the data as shown in Figures. 1 and 2. To begin with, note that the results with $u_{0,L}$ show smaller scaling violations than the results with $u_{0,P}$. The smallest scaling violations are in $u_{0,L}$ at $O(v^6)$, which show little change over a large range of lattice spacing. The scaling analysis provides evidence that $u_{0,L}$ tadpole renormalization yields a more continuum-like action than does $u_{0,P}$.

The most striking feature is the drop in the $b\bar{c}$ splitting at smaller lattice spacings, when $u_{0,P}$ is used at $O(v^6)$. Most c -quark data with $u_{0,P}$ show large changes at small lattice spacings. The $u_{0,L}$ data exhibit much smoother behavior.

We interpret these features as possible indicators of a breakdown in the NRQCD effective action at smaller lattice spacings, when the bare quark mass in lattice units aM_Q^0 falls below one. The bare c -quark mass is larger when $u_{0,L}$ is used.

Another key feature is that the relativistic corrections to the hyperfine splittings are smaller when the action is renormalized using $u_{0,L}$. For example, we find that the charmonium hyperfine splitting is reduced by about 30–40% in going from $O(v^4)$ to $O(v^6)$ when using $u_{0,L}$, compared to a reduction of about 40–60% when using $u_{0,P}$. (Relativistic corrections have been analyzed in the Υ system [14,15], and in heavy-light mesons [16–18]). The $u_{0,P}$ relativistic corrections depend strongly on the lattice spacing and may be related to the pathologies discussed above. This casts new light on the results obtained in Ref. [8], where relativistic corrections to spin splittings in NRQCD were first calculated. The velocity expansion for charmonium may not be as unreliable as was suggested.

We note finally that it is reasonable to attempt to extrapolate the $O(v^6)$ hyperfine splittings for $c\bar{c}$ and $b\bar{c}$ to zero lattice spacing, from the data on coarse lattices, where there is good scaling behavior. However, the extrapolations in the $u_{0,L}$ and $u_{0,P}$ data are clearly very different. This suggests that some relevant operator coefficients c_i in the NRQCD action receive significant $O(\alpha_s)$ corrections. This underlines the need to go beyond tree-level tadpole improvement in order to clarify the differences between renormalization schemes.

We have presented evidence that favors tadpole renormalization using the mean-link in Landau gauge over the fourth root of the average plaquette. This includes a demonstration of better scaling behavior of the hyperfine splittings in three quarkonium systems when $u_{0,L}$ is used, and a smaller size for spin-dependent relativistic corrections. These results help to elucidate the structure of the NRQCD effective action.

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